

DETERMINING THE FIRST ORDER PERTURBATION OF A POLYHARMONIC OPERATOR ON ADMISSIBLE MANIFOLDS

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ABSTRACT. We consider the inverse boundary value problem for the first order perturbation of the polyharmonic operator $\mathcal{L}_{g,X,q}$, with X being a $W^{1,\infty}$ vector field and q being an L^∞ function on compact Riemannian manifolds with boundary which are conformally embedded in a product of the Euclidean line and a simple manifold. We show that the knowledge of the Dirichlet-to-Neumann determines X and q uniquely. The method is based on the construction of complex geometrical optics solutions using the Carleman estimate for the Laplace-Beltrami operator due to Dos Santos Ferreira, Kenig, Salo and Uhlmann. Notice that the corresponding uniqueness result does not hold for the first order perturbation of the Laplace-Beltrami operator.

1. INTRODUCTION

Let (M, g) be a compact oriented Riemannian smooth manifold with boundary. Throughout this paper, the word “smooth” will be used as the synonym of “ C^∞ ”. Let Δ_g be the Laplace-Beltrami operator associated to the metric g which is given in local coordinates by

$$\Delta_g u = |g|^{-1/2} \frac{\partial}{\partial x^j} \left(|g|^{1/2} g^{jk} \frac{\partial u}{\partial x^k} \right),$$

where as usual (g^{jk}) is the matrix inverse of (g_{jk}) , and $|g| = \det(g_{jk})$. If F denotes a function or distribution space (C^k , L^p , H^k , \mathcal{D}' , etc.), then we will denote by $F(M, TM)$ the corresponding space of vector fields on M .

Let $X \in W^{1,\infty}(M, TM)$ and $q \in L^\infty(M)$. Consider the polyharmonic operator $(-\Delta_g)^m$, $m \geq 1$, with the first order perturbation induced by X and q

$$\mathcal{L}_{g,X,q} = (-\Delta_g)^m + X + q$$

The operator $\mathcal{L}_{g,X,q}$ equipped with the domain

$$\mathcal{D}(\mathcal{L}_{g,X,q}) = \{u \in H^{2m}(M) : \gamma u = 0\} = H^{2m}(M) \cap H_0^m(M)$$

is an unbounded closed operator on $L^2(M)$ with purely discrete spectrum; see [8]. Here and in what follows,

$$\gamma u := (u|_{\partial M}, \Delta_g u|_{\partial M}, \dots, \Delta_g^{m-1} u|_{\partial M})$$

is the Dirichlet trace of u , and $H^s(M)$ is the standard Sobolev space on M , $s \in \mathbb{R}$.

We make the assumption that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,X,q}$ in M . Under this assumption, for any $f = (f_0, \dots, f_{m-1}) \in \mathcal{H}_m(\partial M) := \prod_{j=0}^{m-1} H^{2m-2j-1/2}(\partial M)$, the Dirichlet problem

$$\begin{aligned} \mathcal{L}_{g,X,q}u &= 0 \quad \text{in } M, \\ \gamma u &= f \quad \text{in } \partial M, \end{aligned} \tag{1}$$

has a unique solution $u \in H^{2m}(M)$. Let ν be an outer unit normal to ∂M . Introducing the Neumann trace operator $\tilde{\gamma}$ by

$$\begin{aligned} \tilde{\gamma} : H^{2m}(M) &\rightarrow \prod_{j=0}^{m-1} H^{2m-2j-3/2}(\partial M), \\ \tilde{\gamma}u &= (\partial_\nu u|_{\partial M}, \partial_\nu \Delta_g u|_{\partial M}, \dots, \partial_\nu \Delta_g^{m-1} u|_{\partial M}), \end{aligned}$$

we define the Dirichlet-to-Neumann map $N_{g,X,q}$ by

$$N_{g,X,q} : \mathcal{H}_m(\partial M) \rightarrow \prod_{j=0}^{m-1} H^{2m-2j-3/2}(\partial M), \quad N_{g,X,q}(f) = \tilde{\gamma}u,$$

where $u \in H^{2m}(M)$ is the unique solution to the boundary value problem (1). Let us also introduce the set of the Cauchy data for the operator $\mathcal{L}_{g,X,q}$

$$C_{g,X,q} = \{(\gamma u, \tilde{\gamma}u) : u \in H^{2m}(M), \mathcal{L}_{g,X,q}u = 0\}.$$

When 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,X,q}$ in M , the set $C_{g,X,q}$ is the graph of the Dirichlet-to-Neumann map $N_{g,X,q}$.

The inverse problem we are concerned in this paper is to recover the vector field X and the function q from the knowledge of the Dirichlet-to-Neumann map $N_{g,X,q}$ on the boundary ∂M .

When $m = 1$, the Dirichlet-to-Neumann map $N_{g,X,q}$ is invariant under gauge transformations in the following sense. Let ψ be a $C^2(M)$ such that $\psi|_{\partial M} = 0$ and $\partial_\nu \psi|_{\partial M} = 0$. Then

$$e^{-i\psi} \mathcal{L}_{g,X,q} e^{i\psi} = \mathcal{L}_{g,\tilde{X},\tilde{q}}, \quad N_{g,\tilde{X},\tilde{q}} = N_{g,X,q},$$

where

$$\tilde{X} = X + 2\nabla\psi, \quad \tilde{q} = q + \langle X, \nabla\psi \rangle_g + |\nabla\psi|_g^2 - i\Delta_g\psi.$$

Therefore, we may hope to recover X and q from boundary measurements only modulo the above gauge transformations.

In the Euclidean setting, this inverse boundary value problem has been extensively studied, usually in the context of magnetic Schrödinger operators [15, 18, 19, 22, 24]. In the case of Riemannian manifolds, this was proved in [3] for the special class of so-called admissible manifolds.

Let us now introduce admissible manifolds. For this we need the notion of simple manifolds [21]. The notion of simplicity arises naturally in the context of the boundary rigidity problem [17].

Definition 1.1. A compact Riemannian manifold (M, g) with boundary is said to be *simple* if the boundary ∂M is strictly convex, and for any point $x \in M$ the exponential map \exp_x is a diffeomorphism from its maximal domain in $T_x M$ onto M .

Definition 1.2. A compact Riemannian manifold (M, g) with boundary of dimension $n \geq 3$, is said to be *admissible* if it is conformal to a submanifold with boundary of $\mathbb{R} \times (M_0, g_0)$ where (M_0, g_0) is a simple $(n - 1)$ -dimensional manifold.

Examples of admissible manifolds include the following:

1. Bounded domains in Euclidean space, in the sphere minus a point, or in hyperbolic space. In the last two cases, the manifold is conformal to a domain in Euclidean space via stereographic projection.
2. More generally, any domain in a locally conformally flat manifold is admissible, provided that the domain is appropriately small. Such manifolds include locally symmetric 3-dimensional spaces, which have parallel curvature tensor so their Cotton tensor vanishes (see the [3, Appendix B]).
3. Any bounded domain M in \mathbb{R}^n , endowed with a metric which in some coordinates has the form

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$

with $c > 0$ and g_0 simple, is admissible.

4. The class of admissible metrics is stable under C^2 -small perturbations of g_0 .

It was shown in [12] that, in the Euclidean case, the obstruction to uniqueness coming from the gauge equivalence when $m = 1$ can be eliminated by considering operators of higher order. The purpose of this paper is to extend this result for the case of admissible manifolds.

Our main result is as follows.

Theorem 1.3. *Let (M, g) be admissible, and let $m \geq 2$ be an integer. Suppose that $X_1, X_2 \in W^{1,\infty}(\mathbb{R} \times M_0, T(\mathbb{R} \times M_0)) \cap \mathcal{E}'(M, TM)$ and $q_1, q_2 \in L^\infty(M)$ are such that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g, X_1, q_1}$ and $\mathcal{L}_{g, X_2, q_2}$ in M . If $N_{g, X_1, q_1} = N_{g, X_2, q_2}$, then $X_1 = X_2$ and $q_1 = q_2$.*

The key ingredient in the proof of Theorem 1.3 is the construction of complex geometric optics solutions for the operator $\mathcal{L}_{g, X, q}$ with X being a $W^{1,\infty}$ vector field and q an $L^\infty(M)$ function. For this, we use the method of Carleman estimates which is based on the corresponding Carleman estimate for the Laplacian due to Dos Santos Ferreira, Kenig, Salo and Uhlmann [3].

In Theorem 1.3, the condition that $X_1 = X_2 = 0$ on ∂M is needed to extend the vector fields X_1 and X_2 to a slightly larger simple manifold than M while preserving the $W^{1,\infty}$ regularities. When more regularities on X_j and q_j ($j = 1, 2$) are available, we can show a boundary determination result for the vector fields and thus drop such an assumption. This is the following theorem.

Theorem 1.4. *Let (M, g) be admissible, and let $m \geq 2$ be an integer. Suppose that $X_1, X_2 \in C^\infty(M, TM)$ and $q_1, q_2 \in C^\infty(M)$ are such that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g, X_1, q_1}$ and $\mathcal{L}_{g, X_2, q_2}$ in M . If $N_{g, X_1, q_1} = N_{g, X_2, q_2}$, then $X_1 = X_2$ and $q_1 = q_2$.*

Let $\pi : \mathbb{R} \times M_0 \rightarrow M_0$ be the canonical projection $\pi(x_1, x') = x'$. It is interesting to notice that the boundary determination becomes unnecessary if $(\pi(M), g_0)$ is a simple $(n - 1)$ -dimensional manifold and ∂M is connected.

Theorem 1.5. *Let (M, g) be admissible, and let $m \geq 2$ be an integer. Suppose that $X_1, X_2 \in W^{1,\infty}(M, TM)$ and $q_1, q_2 \in L^\infty(M)$ are such that 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g, X_1, q_1}$ and $\mathcal{L}_{g, X_2, q_2}$ in M . Suppose further that $(\pi(M), g_0)$ is a simple $(n - 1)$ -dimensional manifold and ∂M is connected. If $N_{g, X_1, q_1} = N_{g, X_2, q_2}$, then $X_1 = X_2$ and $q_1 = q_2$.*

In the case of Euclidean space, the recovery of a zeroth order perturbation of the biharmonic operator, that is when $m = 2$, has been studied by Isakov [11], where a uniqueness result was obtained, similarly to the case of the Schrödinger operator. The recovery of a first order perturbation of the biharmonic operator from partial data was studied in [13] in a bounded domain, and in [25] in an infinite slab. Higher order operators occur in the areas of physics and geometry such as the study of the Kirchhoff plate equation in the theory of elasticity, and the study of the Paneitz-Branson operator in conformal geometry; for more details see [7].

Finally, we would like to remark that the problem considered in this paper can be viewed as generalization of the Calderón's inverse conductivity problem [1], known also as electrical impedance tomography. In the fundamental paper by Sylvester and Uhlmann [23] it was shown that C^2 conductivities can be uniquely determined from boundary measurements. A corresponding result was proved by Dos Santos Ferreira, Kenig, Salo and Uhlmann [3] in the setting of admissible geometries.

The structure of the paper is as follows. In Section 2 a Carleman estimate is derived for polyharmonic operators based on a similar estimate for the Laplace-Beltrami operator. Section 3 is devoted to the construction of complex geometric optics solutions for the perturbed polyharmonic operator $\mathcal{L}_{g, X, q}$ with X being a $W^{1,\infty}$ vector field and $q \in L^\infty(M)$. Then the proof of Theorem 1.3 is given in Section 4. Attenuated ray transform is the subject of Section 5. In Section 6, we show that the Dirichlet-to-Neumann map determines X on the boundary, this leads to the proof of Theorem 1.4. Finally, the proof of Theorem 1.5 is given in Section 7.

2. CARLEMAN ESTIMATES FOR POLYHARMONIC OPERATORS

Let (M, g) be a Riemannian manifold with boundary. In this section, following [3, 14], we shall use the method of Carleman estimates to construct complex geometric optics solutions for the equation $\mathcal{L}_{g, X, q}u = 0$ in M , with X being a $W^{1,\infty}$ vector field on M and $q \in L^\infty(M)$.

We start by recalling the definition of the Carleman weight for the semiclassical Laplace-Beltrami operator $-h^2\Delta_g$. Let U be an open manifold without boundart

such that $M \subset\subset U$ and let $\varphi \in C^\infty(U, \mathbb{R})$. Consider the conjugated operator

$$P_\varphi = e^{\varphi/h} (-h^2 \Delta_g) e^{-\varphi/h}.$$

Following [3, 14], we say that φ is a limiting Carleman weight for $-h^2 \Delta_g$ in U , if it has non-vanishing differential, and if it satisfies the Poisson bracket condition

$$\{\overline{p_\varphi}, p_\varphi\}(x, \xi) = 0 \quad \text{when } p_\varphi(x, \xi) = 0, \quad (x, \xi) \in T^*M,$$

where p_φ is the semiclassical principal symbol of P_φ .

First we shall derive a Carleman estimate for the semiclassical polyharmonic operator $(-h^2 \Delta_g)^m$, where $h > 0$ is a small parameter, by iterating the corresponding Carleman estimate for the semiclassical Laplace-Beltrami operator $-h^2 \Delta_g$, which we now proceed to recall the following [3, 14].

We use the notation $d\text{Vol}_g$ for the volume form of (M, g) . For any two functions u, v on M , define an inner product

$$(u|v) := \int_M u(x) \overline{v(x)} d\text{Vol}_g(x),$$

and the corresponding norm will be denoted by $\|\cdot\|_{L^2(M)}$. We also write for short

$$\|\nabla u\|_{L^2(M)} = \|\nabla u\|_{L^2(M)} = \left(\int_M |\nabla u(x)|_g^2 d\text{Vol}_g(x) \right)^{1/2}.$$

We assume that (M, g) is embedded in a compact manifold (N, g) without boundary, and φ is a limiting Carleman weight on (U, g) , where U is an open submanifold of N such that $M \subset\subset U$. By semiclassical spectral theorem one can define for $s \in \mathbb{R}$ the semiclassical Bessel potentials $J^s = (1 - h^2 \Delta_g)^{s/2}$. One has $J^s J^t = J^{s+t}$, and Bessel potentials commute with any function of $-\Delta_g$. Define for $s \in \mathbb{R}$ the semiclassical Sobolev space associated to the norm

$$\|u\|_{H_{\text{scl}}^s(N)} = \|J^s u\|_{L^2(N)}.$$

Our starting point is the following Carleman estimate for the semiclassical Laplace-Beltrami operator $-h^2 \Delta_g$ which is due to Dos Santos Ferreira, Kenig, Salo and Uhlmann [3, Lemma 4.3]. In what follows, $A \lesssim B$ means that $A \leq CB$ where $C > 0$ is a constant independent of h and A, B .

Proposition 2.1. *Let (U, g) be an open Riemannian manifold and (M, g) be a smooth compact Riemannian submanifold with boundary such that $M \subset\subset U$. Let φ be a limiting Carleman weight on (U, g) . Then for all $h > 0$ small enough and $s \in \mathbb{R}$, we have*

$$h \|u\|_{H_{\text{scl}}^{s+1}(N)} \lesssim \|e^{\varphi/h} (-h^2 \Delta_g) e^{-\varphi/h} u\|_{H_{\text{scl}}^s(N)}$$

for all $u \in C_0^\infty(M)$.

Next we shall derive a Carleman estimate for the operator $\mathcal{L}_{g,X,q}$ with X being a $W^{1,\infty}$ vector field on M and $q \in L^\infty(M)$. To that end we shall use Proposition 2.1 with $s = -1$. We have the following result.

Proposition 2.2. *Let (U, g) be an open Riemannian manifold and (M, g) be a smooth compact Riemannian submanifold with boundary such that $M \subset\subset U$. Let φ be a limiting Carleman weight on (U, g) . Suppose that X is a $W^{1,\infty}$ vector field on M and $q \in L^\infty(M)$. Then for all $h > 0$ small enough, we have*

$$\|u\|_{L^2(N)} \lesssim \frac{1}{h^m} \|e^{\varphi/h}(h^{2m}\mathcal{L}_{g,X,q})e^{-\varphi/h}u\|_{H_{\text{scl}}^{-1}(N)}, \quad (2)$$

for all $u \in C_0^\infty(M)$.

Proof. Iterating the Carleman estimate in Proposition 2.2 m times, $m \geq 2$, we get the following Carleman estimate for the polyharmonic operator,

$$h^m \|u\|_{H_{\text{scl}}^{s+m}(N)} \lesssim \|e^{\varphi/h}(-h^2\Delta_g)^m e^{-\varphi/h}u\|_{H_{\text{scl}}^s(N)},$$

for all $u \in C_0^\infty(\Omega)$, $s \in \mathbb{R}$ and $h > 0$ small enough. We shall use this estimate with $s = -1$:

$$h^m \|u\|_{H_{\text{scl}}^{m-1}(N)} \lesssim \|e^{\varphi/h}(-h^2\Delta_g)^m e^{-\varphi/h}u\|_{H_{\text{scl}}^{-1}(N)}, \quad (3)$$

for all $u \in C_0^\infty(N)$ and $h > 0$ small enough. Since we are dealing with first order perturbations of the polyharmonic operator and $m \geq 2$, the following weakened version of (3) will be sufficient for our purposes

$$h^m \|u\|_{L^2(N)} \lesssim \|e^{\varphi/h}(-h^2\Delta_g)^m e^{-\varphi/h}u\|_{H_{\text{scl}}^{-1}(N)}, \quad (4)$$

for all $u \in C_0^\infty(M)$ and $h > 0$ small enough.

It is easy to see that

$$\|e^{\varphi/h}h^{2m}qe^{-\varphi/h}u\|_{L^2(N)} \lesssim h^{2m}\|q\|_{L^\infty(M)}\|u\|_{H_{\text{scl}}^1(N)}. \quad (5)$$

Note that $e^{\varphi/h}h^{2m}X(e^{-\varphi/h}u) = -h^{2m-1}\langle X, \nabla\varphi \rangle_g u + h^{2m}\langle X, \nabla u \rangle_g$. Therefore, since $m \geq 2$

$$\begin{aligned} \|h^{2m-1}\langle X, \nabla\varphi \rangle_g u\|_{L^2(N)} &\leq h^{2m-1}\|\langle X, \nabla\varphi \rangle_g\|_{L^\infty(M)}\|u\|_{L^2(N)} \\ &\leq h^m\|\langle X, \nabla\varphi \rangle_g\|_{L^\infty(M)}\|u\|_{H_{\text{scl}}^1(N)} \end{aligned}$$

and

$$\|h^{2m}\langle X, h\nabla u \rangle_g\|_{L^2(N)} \leq h^{2m-1}\|X\|_{L^\infty(M)}\|u\|_{H_{\text{scl}}^1(N)} \leq h^m\|X\|_{L^\infty(M)}\|u\|_{H_{\text{scl}}^1(N)}.$$

imply

$$\|e^{\varphi/h}h^{2m}X(e^{-\varphi/h}u)\|_{L^2(N)} \lesssim h^m\|u\|_{H_{\text{scl}}^1(N)}.$$

Combining this together with estimates (4) and (5), we get the result. \square

Set

$$\mathcal{L}_\varphi := e^{\varphi/h}(h^{2m}\mathcal{L}_{g,X,q})e^{-\varphi/h}.$$

Then we have

$$\langle \mathcal{L}_\varphi u, \bar{v} \rangle_\Omega = \langle u, \overline{\mathcal{L}_\varphi^* v} \rangle_\Omega, \quad u, v \in C_0^\infty(\Omega),$$

where $\mathcal{L}_\varphi^* = e^{-\varphi/h}(h^2\mathcal{L}_{g,-X,-\text{div}_g X+q})e^{\varphi/h}$ is the formal adjoint of \mathcal{L}_φ , and $\langle \cdot, \cdot \rangle_M$ is the distribution duality on M . The estimate in Proposition 2.2 holds for \mathcal{L}_φ^* , since $-\varphi$ is a limiting Carleman weight as well.

To construct the complex geometric optics solutions for the operator $\mathcal{L}_{g,X,q}$, we need to convert the Carleman estimate (2) for \mathcal{L}_φ^* into the following solvability result.

The proof is essentially well-known, and we include it here for the convenience of the reader. We shall use the following notation for the semiclassical Sobolev norm on M

$$\|u\|_{H_{\text{scl}}^1(M)}^2 = \|u\|_{L^2(M)}^2 + \|h\nabla u\|_{L^2(M)}^2.$$

Proposition 2.3. *Let X be a $W^{1,\infty}$ vector field on M and $q \in L^\infty(M)$ and assume that $m \geq 2$. If $h > 0$ is small enough, then for any $v \in L^2(M)$ there is a solution $u \in H^1(M)$ of the equation*

$$e^{\varphi/h} h^{2m} \mathcal{L}_{g,X,q} e^{-\varphi/h} u = v$$

satisfying

$$\|u\|_{H_{\text{scl}}^1(M)} \leq \frac{C}{h^m} \|v\|_{L^2(M)}.$$

Proof. Let $v \in H^{-1}(M)$ and let us consider the following complex linear functional,

$$L : \mathcal{L}_\varphi^* C_0^\infty(M) \rightarrow \mathbb{C}, \quad \mathcal{L}_\varphi^* w \mapsto \langle w, \bar{v} \rangle_M.$$

By the Carleman estimate (2) for \mathcal{L}_φ^* , the map L is well-defined. Let $w \in C_0^\infty(M)$. Then we have

$$\begin{aligned} |L(\mathcal{L}_\varphi^* w)| &= |\langle w, \bar{v} \rangle_M| \leq \|w\|_{L^2(N)} \|v\|_{L^2(M)} \\ &\lesssim \frac{1}{h^m} \|v\|_{L^2(M)} \|\mathcal{L}_\varphi^* w\|_{H_{\text{scl}}^{-1}(N)}. \end{aligned}$$

By the Hahn-Banach theorem, we may extend L to a linear continuous functional \tilde{L} on $L^2(N)$, without increasing its norm. By the Riesz representation theorem, there exists $u \in H^1(\mathbb{R}^n)$ such that for all $\psi \in H^{-1}(\mathbb{R}^n)$,

$$\tilde{L}(\psi) = \langle \psi, \bar{u} \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \|u\|_{H_{\text{scl}}^1(\mathbb{R}^n)} \lesssim \frac{1}{h^m} \|v\|_{L^2(\Omega)}.$$

Let us now show that $\mathcal{L}_\varphi u = v$ in Ω . To that end, let $w \in C_0^\infty(\Omega)$. Then

$$\langle \mathcal{L}_\varphi u, \bar{w} \rangle_\Omega = \langle u, \overline{\mathcal{L}_\varphi^* w} \rangle_{\mathbb{R}^n} = \overline{\tilde{L}(\mathcal{L}_\varphi^* w)} = \overline{\langle w, \bar{v} \rangle_\Omega} = \langle v, \bar{w} \rangle_\Omega.$$

The proof is complete. \square

3. COMPLEX GEOMETRIC OPTICS SOLUTIONS

Let φ be a limiting Carleman weight in an admissible manifold (M, g) . We will construct solutions to $\mathcal{L}_{g,X,q} u = 0$ in M of the form

$$u = e^{-(\varphi+i\psi)/h} (a + r), \tag{6}$$

where a is an amplitude, r is a correction term which is small when $h > 0$ is small, and ψ is a real valued phase.

Set $\rho = \varphi + i\psi$ for the complex valued phase. Consider the conjugated operator $P_\rho = e^{\rho/h} h^{2m} \mathcal{L}_{g,X,q} e^{-\rho/h}$, which has the following expression

$$P_\rho = (-h^2 \Delta_g - |\nabla \rho|_g^2 + h \Delta_g \rho + 2h \nabla \rho)^m + h^{2m} X - h^{2m-1} \langle -\nabla \rho, X \rangle_g + h^{2m} q.$$

Here and in what follows, the norm $|\cdot|_g^2$ and the inner product $\langle \cdot, \cdot \rangle_g$ are extended to complex valued tangent vectors by

$$\langle \zeta, \eta \rangle_g = \langle \operatorname{Re} \zeta, \operatorname{Re} \eta \rangle_g - \langle \operatorname{Im} \zeta, \operatorname{Im} \eta \rangle_g + i(\langle \operatorname{Re} \zeta, \operatorname{Im} \eta \rangle_g + \langle \operatorname{Im} \zeta, \operatorname{Re} \eta \rangle_g), \quad |\zeta|_g^2 = \langle \zeta, \zeta \rangle_g.$$

Since $m \geq 2$, in order to get

$$e^{\varphi/h} h^{2m} \mathcal{L}_{g,X,q}(e^{-\varphi/h} a) = \mathcal{O}(h^{m+1}),$$

in $L^2(M)$, we should choose ρ satisfying the following eikonal equation

$$|\nabla \rho|_g^2 = 0 \quad \text{in } M, \tag{7}$$

and choose $a \in C^\infty(M)$ satisfying the following transport equation

$$(2\nabla \rho + \Delta_g)^m a = 0 \quad \text{in } M. \tag{8}$$

Recall that (M, g) is conformally embedded in $\mathbb{R} \times (M_0, g_0)$, where (M_0, g_0) is some simple $(n-1)$ -dimensional manifold. If necessary, we replace M_0 with a slightly larger simple manifold. Therefore, we can and shall assume that for some simple $(D, g_0) \subset \subset (M_0^{\text{int}}, g_0)$ one has

$$(M, g) \subset \subset (\mathbb{R} \times D^{\text{int}}, g) \subset (\mathbb{R} \times M_0^{\text{int}}, g). \tag{9}$$

Note that $\mathbb{R} \times M_0$ has global coordinate chart in which the metric g has the following form

$$g(x) = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \tag{10}$$

where $c > 0$ and g_0 is simple. A natural choice of the limiting Carleman weight is $\varphi(x) = x_1$. Then the equation (7) for the complex valued phase ρ becomes

$$|\nabla \psi|^2 = \frac{1}{c}, \quad \partial_{x_1} \psi = 0.$$

This equation will be solved using special coordinates on (M, g) . This is based on the so-called polar coordinates on the transversal simple manifold (M_0, g_0) . Let $\omega \in D$ be such that $(x_1, \omega) \notin M$ for all x_1 . Points of M have the form $x = (x_1, r, \theta)$ where (r, θ) are polar normal coordinates in (D, g_0) with center ω . That is, $x' = \exp_\omega^D(r\theta)$ where $r > 0$ and $\theta \in S^{n-2}$. In terms of these coordinates the metric g has the form

$$g(x_1, r, \theta) = c(x_1, r, \theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m(r, \theta) \end{pmatrix},$$

where m is a smooth positive definite matrix.

We solve (7) by simply taking $\psi(x) = \psi_\omega(x) = r$. Thus, the complex valued phase has the form $\rho = x_1 + ir$ and its gradient is $\nabla \rho = \frac{2}{c} \bar{\partial}$, where

$$\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial r} \right).$$

Next, we solve transport equation (8). In the coordinates (x_1, r, θ) equation (8) becomes

$$\left(\frac{4}{c} \bar{\partial} + \frac{1}{c} \log \frac{|g|}{c^2} \right)^m a = 0.$$

Consider a as the function having the following form

$$a = |g|^{-1/4} c^{1/2} a_0(x_1, r, \theta) b(\theta)$$

where b is smooth and a_0 is such that $\bar{\partial}a_0 = ca_1$ for some a_1 satisfying $\bar{\partial}a_1 = 0$. Note that (6) will be a solution for $\mathcal{L}_{g,X,q}u = 0$ if $P_\rho(a + hr) = 0$. Then, with the choice of φ and ψ made above, this equation is equivalent to the following

$$e^{\varphi/h} h^{2m} \mathcal{L}_{g,X,q} e^{-\varphi/h} (e^{-i\psi/h} hr) = -e^{-i\psi/h} (h^{2m} \mathcal{L}_{g,X,q} a + h^{2m-1} \langle \nabla \rho, X \rangle_g a).$$

This will be solved by using Proposition 2.3. We find $r \in H^1(M)$ satisfying

$$\|r\|_{H_{\text{scl}}^1(M)} = \mathcal{O}(1).$$

The discussion of this section can be summarized in the following proposition.

Proposition 3.1. *Assume that (M, g) satisfies (9) and (10), and let $m \geq 2$ be an integer. Suppose that X is a $W^{1,\infty}$ vector field on M and $q \in L^\infty(M)$. Let $\omega \in D$ such that $(x_1, \omega) \notin M$ for all x_1 . If (r, θ) are polar normal coordinates in (D, g_0) with center ω , then the equation*

$$\mathcal{L}_{g,X,q} u = 0 \quad \text{in } M$$

has a solution of the form

$$u = e^{-\frac{1}{h}(\varphi+i\psi)} (|g|^{-1/4} c^{1/2} a_0(x_1, r, \theta) b(\theta) + hr),$$

where $\bar{\partial}a_0 = ca_1$ for some a_1 depending on (x_1, r) and satisfying $\bar{\partial}a_1 = 0$, b is smooth and the remainder term $r \in H^1(M)$ such that $\|r\|_{H_{\text{scl}}^1(M)} = \mathcal{O}(1)$.

Remark 3.2. In fact, we need complex geometric optics solutions belonging to $H^{2m}(M)$. Such solutions can be obtained in the following way. Extend X and q smoothly to $\mathbb{R} \times M_0$. By elliptic regularity, the complex geometric optics solutions constructed as above in M_0 will belong to $H^{2m}(M)$.

Remark 3.3. It is easy to check that if a_0 depends only on (x_1, r) and satisfies $\bar{\partial}a_0 = 0$, then the equation $\mathcal{L}_{g,X,q}u = 0$ in M has a solution as in Proposition 3.1.

4. PROOF OF THEOREM 1.3

Let (M, g) be an admissible manifold and let $m \geq 2$ be an integer. The first ingredient in the proof of Theorem 1.3 is a standard reduction to a larger compact manifold with boundary.

Proposition 4.1. *Let M, M_1 be compact manifolds with boundary such that $M \subset\subset M_1$, and let $m \geq 2$ be an integer. Assume that X_1, X_2 are $W^{1,\infty}$ vector fields on M and $q_1, q_2 \in L^\infty(M)$. Suppose that*

$$X_1 = X_2, \quad q_1 = q_2 \quad \text{in } M_1 \setminus M.$$

If $C_{g,X_1,q_1}^M = C_{g,X_2,q_2}^M$, then $C_{g,X_1,q_1}^{M_1} = C_{g,X_2,q_2}^{M_1}$, where $C_{g,X_j,q_j}^{M_1}$ denotes the set of the Cauchy data for \mathcal{L}_{g,X_j,q_j} in M_1 , $j = 1, 2$.

Proof. Let $u \in H^{2m}(M_1)$ be a solution of $\mathcal{L}_{g,X_1,q_1}u = 0$ in M_1 . Since $C_{g,X_1,q_1}^M = C_{g,X_2,q_2}^M$, there exists $v \in H^{2m}(M)$, solving $\mathcal{L}_{g,X_2,q_2}v = 0$ in M , and satisfying $\gamma v = \gamma u$ in ∂M and $\tilde{\gamma}v = \tilde{\gamma}u$ in ∂M . Setting

$$v_1 = \begin{cases} v & \text{in } M, \\ u & \text{in } M_1 \setminus M, \end{cases}$$

we get $v_1 \in H^{2m}(M_1)$ and $\mathcal{L}_{g,X_2,q_2}v_1 = 0$ in M_1 . Thus, $C_{g,X_1,q_1}^{M_1} \subset C_{g,X_2,q_2}^{M_1}$. Exactly the same way but in the other direction finishes the proof. \square

The second ingredient is the derivation of the following integral identity based on the assumption that $C_{g,X_1,q_1}^M = C_{g,X_2,q_2}^M$.

Proposition 4.2. *Let (M, g) be a compact Riemannian manifold with boundary, and let $m \geq 2$ be an integer. Assume that X_1, X_2 are $W^{1,\infty}$ vector fields on M and $q_1, q_2 \in L^\infty(M)$. If $C_{g,X_1,q_1} = C_{g,X_2,q_2}$, then*

$$\int_M [\langle X_1 - X_2, v \nabla u \rangle_g + (q_1 - q_2)uv] d\text{Vol}_g(x) = 0,$$

for any $u, v \in H^{2m}(M)$ satisfying $\mathcal{L}_{g,-X_1,-\text{div}_g X_1+q_1}v = 0$ and $\mathcal{L}_{g,X_2,q_2}u = 0$ in M .

Proof. We will use the following consequence of the Green's formula, see [8],

$$(\mathcal{L}_{g,X_1,q_1}u, v)_{L^2(M)} = (u, \mathcal{L}_{g,X_1,q_1}^*v)_{L^2(M)} \quad (11)$$

for all $u, v \in H^{2m}(M)$ such that $\gamma u = \gamma v = 0$, where $\mathcal{L}_{g,X_1,q_1}^* = \mathcal{L}_{g,-X_1,-\text{div}_g X_1+q_1}$.

Now, let $u, v \in H^{2m}(M)$ be such that $\mathcal{L}_{g,-X_1,-\text{div}_g X_1+q_1}v = 0$ and $\mathcal{L}_{g,X_2,q_2}u = 0$ in M . The hypothesis that $\mathcal{N}_{g,X_1,q_1} = \mathcal{N}_{g,X_2,q_2}$ implies the existence of $\tilde{u} \in H^{2m}(M)$ such that $\mathcal{L}_{g,X_1,q_1}\tilde{u} = 0$ and $\gamma\tilde{u} = \gamma u$, $\tilde{\gamma}\tilde{u} = \tilde{\gamma}u$. We have

$$\mathcal{L}_{g,X_1,q_1}(u - \tilde{u}) = (X_1 - X_2)\tilde{u} + (q_1 - q_2)\tilde{u}.$$

Using (11), this implies the result. \square

According to hypothesis, that $X_1 = X_2$ in $(\mathbb{R} \times M_0) \setminus M^{\text{int}}$. We also extend q_1 and q_2 to $\mathbb{R} \times M_0$ by zero outside M^{int} . Let, as in Section 3, $(D, g_0) \subset \subset (M_0^{\text{int}}, g_0)$ be simple such that $(M, g) \subset \subset (\mathbb{R} \times D^{\text{int}}, g) \subset (\mathbb{R} \times M_0^{\text{int}}, g)$. Let (M_1, g) be also admissible and simply connected such that $(M, g) \subset \subset (M_1^{\text{int}}, g)$ and $(M_1, g) \subset \subset (\mathbb{R} \times D^{\text{int}}, g)$. According to Proposition 4.1, we know that $C_{g,X_1,q_1}^{M_1} = C_{g,X_2,q_2}^{M_1}$ is true.

According to Proposition 4.2 the following integral identity holds for all $u, v \in H^{2m}(M_1)$ satisfying $\mathcal{L}_{g,X_2,q_2}u = 0$ and $\mathcal{L}_{g,-X_1,-\text{div}_g X_1+q_1}v = 0$ in M_1 , respectively:

$$\int_{M_1} [\langle X_1 - X_2, v \nabla u \rangle_g + (q_1 - q_2)uv] d\text{Vol}_g(x) = 0. \quad (12)$$

The main idea of the proof of Theorem 1.3 is to use the integral identity (12) with $u, v \in H^{2m}(M)$ being complex geometric optics solutions for the equations

$\mathcal{L}_{g, X_2, q_2} u = 0$ and $\mathcal{L}_{g, -X_1, -\operatorname{div}_g X_1 + q_1} v = 0$ in M_1 , respectively. We use Proposition 3.1, Remark 3.2 and Remark 3.3 to choose solutions of the form

$$\begin{aligned} u &= e^{-\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + hr_1), \\ v &= e^{\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2} + hr_2), \end{aligned}$$

where $\lambda \in \mathbb{R}$ and $\|r_j\|_{H_{\text{scl}}^1(M_1)} = \mathcal{O}(1)$, $j = 1, 2$. Substituting these solutions in (12), multiplying the resulting equality by h and letting $h \rightarrow 0$, we get

$$\lim_{h \rightarrow 0} \int_{M_1} \langle X_1 - X_2, \nabla \rho \rangle_g uv d\operatorname{Vol}_g(x) = 0,$$

where $\rho = x_1 + ir$. Let us rewrite the integral in (x_1, r, θ) coordinates. Write $X = X_1 - X_2$, and let X^\flat be a 1-form dual to X . Let $X_{x_1}^\flat$ and X_r^\flat denote the components of X^\flat in the x_1 and r coordinates. Then

$$\int_{\mathbb{R}} \int_{M_{1,x_1}} (X_{x_1}^\flat + iX_r^\flat) e^{i\lambda(x_1+ir)} b(\theta) dr d\theta dx_1 = 0, \quad (13)$$

where $M_{1,x_1} = \{(r, \theta) : (x_1, r, \theta) \in M_1\}$. Since $X_1 = X_2$ in $(\mathbb{R} \times M_0) \setminus M^{\text{int}}$, we may assume that the integral is over $\mathbb{R} \times D$. Taking x_1 -integral inside gives

$$\int_{S^{n-2}} \int_{\mathbb{R}} e^{-\lambda r} \left(\int_{\mathbb{R}} e^{i\lambda x_1} (X_{x_1}^\flat + iX_r^\flat)(x_1, r, \theta) dx_1 \right) dr d\theta = 0.$$

Define

$$f(x') = \int_{\mathbb{R}} e^{i\lambda x_1} X_{x_1}^\flat(x_1, x') dx_1, \quad \alpha(x') = \sum_{j=2}^n \left(\int_{\mathbb{R}} e^{i\lambda x_1} X_j^\flat(x_1, x') dx_1 \right) dx^j.$$

Then $f \in W^{1,\infty}(D)$ and α is a 1-form which is $W^{1,\infty}$ on D , and the integral identity above can be rewritten as

$$\int_{S^{n-2}} \int_{\mathbb{R}} e^{-\lambda r} [f(\gamma_{w,\theta}(r)) + i\alpha(\dot{\gamma}_{w,\theta}(r))] dr d\theta = 0,$$

where $\gamma_{w,\theta}$ is a geodesic in (D, g_0) issued from the point w in the direction θ . For $w \in \partial D$, the integral above is related to the attenuated ray transform of function f and 1-form $i\alpha$ in D with constant attenuation $-\lambda$. Therefore, by varying the point w in Proposition 3.1 on ∂D and using Proposition 5.1 in Section 5, for small enough λ , we have $f = -\lambda p$ and $\alpha = -idp$ where $p \in W^{1,\infty}(D)$ and $p|_{\partial D} = 0$. The definition of α and analyticity of the Fourier transform imply that

$$\partial_k X_j^\flat - \partial_j X_k^\flat = 0, \quad j, k = 2, \dots, n.$$

Also

$$\int e^{i\lambda x_1} (\partial_j X_1^\flat - \partial_1 X_j^\flat)(x_1, x') dx_1 = \partial_j f + i\lambda \alpha_j = 0,$$

showing that $dX^\flat = 0$ in M_1 . Since M_1 is simply connected, there is $\phi \in W^{2,\infty}(M_1)$ such that $\phi|_{\partial M_1} = 0$ and $X = \nabla\phi$.

Since $X = X_1 - X_2$ in the neighborhood of the boundary ∂M_1 , we conclude that ϕ is a constant, say $c \in \mathbb{C}$, on ∂M_1 . Therefore, considering $\phi - c$, we may and will

assume that $\phi = 0$ on ∂M_1 . Since $X_1 = X_2$ in $(\mathbb{R} \times M_0) \setminus M^{\text{int}}$, we also may and shall assume that ϕ is zero outside M_1 . In particular, ϕ is compactly supported.

Next, we show that $X_1 = X_2$. For this, using Proposition 3.1 and Remark 3.2, consider

$$\begin{aligned} u &= e^{-\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + hr_1), \\ v &= e^{\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2}a_0 + hr_2), \end{aligned}$$

where a_0 satisfies $\bar{\partial}a_0 = c$. Such a_0 can be constructed using Cauchy's integral formula in [6] as

$$a_0(x_1, r, \theta) = a_0(\rho, \theta) = \frac{1}{2\pi} \int_{\mathcal{B}} \frac{c(z, \theta)}{z - \rho} dz \wedge d\bar{z}, \quad \text{for all } \theta \in S^{n-2},$$

where $\rho = x_1 + ir$, \mathcal{B} is a bounded domain in the upper half plane $\mathbb{H} \subset \mathbb{C}$ such that the map $\mathcal{B} \times S^{n-2} \rightarrow \mathbb{R} \times M_0$, $(x_1, r, \theta) \mapsto (x_1, \exp_{\omega}^D(r\theta))$ covers M_1 and the boundary $\partial\mathcal{B}$ is piecewise smooth. Here and in what follows, $\omega \in D$ such that $\omega \in M_1$ in Proposition 3.1.

Substituting these solutions and $X_1 - X_2 = \nabla\phi$ in (12), multiplying the resulting equality by h and letting $h \rightarrow 0$, we get

$$\lim_{h \rightarrow 0} \int_{M_1} \langle \nabla\phi, \nabla\rho \rangle_g uv d\text{Vol}_g(x) = 0,$$

where $\rho = x_1 + ir$. Rewriting the integral in (x_1, r, θ) coordinates and taking x_1 -integral inside, we obtain

$$2 \int_{S^{n-2}} \left(\int_0^\infty \int_{\mathbb{R}} \bar{\partial}\phi a_0 e^{i\lambda(x_1+ir)} b(\theta) dx_1 dr \right) d\theta = 0.$$

Since ϕ is compactly supported, integrating by parts, in (x_1, r) , gives

$$\begin{aligned} 0 &= - \int_{S^{n-2}} \left(\int_0^\infty \int_{\mathbb{R}} \bar{\partial}\phi a_0 e^{i\lambda(x_1+ir)} b(\theta) dx_1 dr \right) d\theta \\ &= \int_{S^{n-2}} \left(\int_0^\infty \int_{\mathbb{R}} \phi \bar{\partial}a_0 e^{i\lambda(x_1+ir)} b(\theta) dx_1 dr \right) d\theta \\ &= \int_{S^{n-2}} \left(\int_0^\infty \int_{\mathbb{R}} \phi c e^{i\lambda(x_1+ir)} b(\theta) dx_1 dr \right) d\theta. \end{aligned} \tag{14}$$

Set

$$\Phi_\lambda(r, \theta) = \int_{\mathbb{R}} \phi c e^{i\lambda x_1} dx_1,$$

i.e. Φ_λ is the Fourier transform of ϕc in x_1 -variable. Then (14) can be written as

$$\int_{S^{n-2}} \int e^{-\lambda r} \Phi_\lambda(\gamma_{\omega, \theta}(r)) b(\theta) dr d\theta = 0.$$

By varying the point ω in Proposition 3.1 on ∂D and using [5, Lemma 5.1], for small enough λ , we have $\Phi_\lambda = 0$. Since ϕc is compactly supported, its Fourier transform Φ_λ is analytic. Therefore we obtain, $\phi = 0$ which shows that $X_1 = X_2$.

To show that $q_1 = q_2$, consider (12) with $X_1 = X_2$ which becomes

$$\int_{M_1} (q_1 - q_2)uv \, d\text{Vol}_g(x) = 0 \quad (15)$$

holds for all $u, v \in H^{2m}(M_1)$ satisfying $\mathcal{L}_{g, X_2, q_2} u = 0$ and $\mathcal{L}_{g, -X_1, -\text{div}_g X_1 + q_1} v = 0$ in M_1 , respectively. Use Proposition 3.1, Remark 3.2 and Remark 3.3 to choose solutions of the form

$$\begin{aligned} u &= e^{-\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + hr_1), \\ v &= e^{\frac{1}{h}(x_1+ir)}(|g|^{-1/4}c^{1/2} + hr_2), \end{aligned}$$

where $\lambda \in \mathbb{R}$ and $\|r_j\|_{H^1_{\text{sel}}(M_1)} = \mathcal{O}(1)$, $j = 1, 2$. Substituting these solutions in (15) and letting $h \rightarrow 0$, we get

$$\int_{\mathbb{R}} \int_{M_{1,x_1}} e^{i\lambda(x_1+ir)}(q_1 - q_2)c(x_1, r, \theta)b(\theta) \, dr \, d\theta \, dx_1 = 0.$$

Taking x_1 -integral inside and varying b gives

$$\int_{S^{n-2}} \int_0^\infty \int_{\mathbb{R}} e^{i\lambda(x_1+ir)}(q_1 - q_2)c(x_1, r, \theta) \, dx_1 \, dr \, d\theta = 0.$$

Set

$$Q_\lambda(r, \theta) = \int_{\mathbb{R}} (q_1 - q_2)c e^{i\lambda x_1} \, dx_1,$$

i.e. Q_λ is the Fourier transform of $(q_1 - q_2)c$ in x_1 -variable. Then, as in the case of Φ_λ , one can show that $Q_\lambda = 0$ for all λ small enough. We have extended q_1 and q_2 to $\mathbb{R} \times M_0$ by zero outside M , which implies that $q_1 - q_2$ is compactly supported. Hence, Q_λ is analytic. This together with $Q_\lambda = 0$ for all λ small enough, allows us to conclude that $q_1 = q_2$.

5. ATTENUATED RAY TRANSFORM

The aim of this section is to prove the following proposition which was used in the proof of Theorem 1.3. We will closely follow the arguments in [5].

Proposition 5.1. *Let (D, g_0) be an $(n - 1)$ -dimensional simple manifold. Let $f \in L^\infty(D)$ and α be a 1-form which is L^∞ on D . Consider the integrals*

$$\int_{S^{n-2}} \int_0^{\tau(\omega, \theta)} [f(\gamma_{\omega, \theta}(r)) + \alpha_k(\gamma_{\omega, \theta}(r))\dot{\gamma}_{\omega, \theta}^k(r)]e^{-\lambda r}b(\theta) \, dr \, d\theta,$$

where (r, θ) are polar normal coordinates in (D, g_0) centered at some $\omega \in \partial D$, and $\tau(\omega, \theta)$ is the time when the geodesic $r \mapsto (r, \theta)$ exits D . If $|\lambda|$ is sufficiently small, and if these integrals vanish for all $\omega \in \partial D$ and all $b \in C^\infty(S^{n-2})$, then there is $p \in W^{1,\infty}(D)$ with $p|_{\partial D} = 0$ such that $f = -\lambda p$ and $\alpha = dp$.

This is related to the injectivity of attenuated ray transform acting on function and 1-form on D . Let us introduce some notions and facts; see [21] for more details. By SD we will denote its unit sphere bundle $SD := \{(x, v) \in TD : |v|_{g_0(x)} = 1\}$. On

the boundary of D , we consider the set of inward and outward unit vectors defined as

$$\begin{aligned}\partial_+SD &= \{(x, v) \in SD : x \in \partial D, \langle v, \nu(x) \rangle_{g_0(x)} \leq 0\}, \\ \partial_-SD &= \{(x, v) \in SD : x \in \partial D, \langle v, \nu(x) \rangle_{g_0(x)} \geq 0\},\end{aligned}$$

where ν is the unit outer normal to ∂D . The geodesics entering D can be parameterized by ∂_+SD . For any $(x, v) \in SD$ the first non-negative exit time of the geodesic $\gamma_{x,v}$, with $x = \gamma_{x,v}(0)$, $v = \dot{\gamma}_{x,v}(0)$, will be denoted as $\tau(x, v)$. Simplicity assumption guarantees that $\tau(x, v)$ is finite for all $(x, v) \in SD$. We also write $\phi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$ for the geodesic flow.

We endow the unit sphere bundle SD with its usual Liouville (local product) measure $d\Sigma^{2n-3}$, and endow the bundle ∂_+SD with its standard measure $d\Sigma^{2n-4}$. By $d\sigma_x$ we denote the measure on S_xD .

Let f be a function and α be a 1-form on D . The geodesic ray transform of f and α , with constant attenuation $-\lambda$, is defined as

$$T_\lambda[f, \alpha](x, v) = \int_0^{\tau(x, v)} [f(\gamma_{x,v}(t)) + \alpha_k(\gamma_{x,v}(t))\dot{\gamma}_{x,v}^k(t)]e^{-\lambda t} dt, \quad (x, v) \in \partial_+SD.$$

In Proposition 5.1, if f and α were a continuous function and 1-form, respectively, one could choose $b(\theta)$ to approximate a delta function at fixed angles θ and obtain that

$$\int_0^{\tau(\omega, \theta)} [f(\gamma_{\omega, \theta}(r)) + \alpha_k(\gamma_{\omega, \theta}(r))\dot{\gamma}_{\omega, \theta}^k(r)]e^{-\lambda r} dr = 0$$

for all $\omega \in \partial D$ and all $\theta \in S^{n-2}$. This would imply that

$$T_\lambda[f, \alpha](x, v) = 0 \quad \text{for all } (x, v) \in \partial_+SD.$$

We will use the following result from [3, Theorem 7.1].

Proposition 5.2. *Let (D, g_0) be a compact simple manifold with smooth boundary. There exists $\varepsilon > 0$ such that the following assertion holds for a real number λ with $|\lambda| < \varepsilon$: If $f \in C^\infty(D)$ and α be a smooth 1-form on D , then $T_\lambda[f, \alpha](x, v) = 0$ for all $(x, v) \in \partial_+SD$ implies the existence of $p \in C^\infty(D)$ with $p|_{\partial D} = 0$ such that $f = -\lambda p$ and $\alpha = dp$.*

The previous argument together with the above theorem proves Proposition 5.1 for smooth f and α . However, this requires f and α to be C^∞ -smooth in D and it is not obvious how to do this when f and α are L^∞ on D . We resolve this problem by using duality and the ellipticity of the normal operator $T_\lambda^*T_\lambda$.

In the space of functions on ∂_+SD define the inner product

$$(h, h')_{L_\mu^2(\partial_+SD)} := \int_D h h' d\mu$$

where $d\mu(x, v) = \langle v, \nu \rangle_{g_0(x)} d\Sigma^{2n-4}$. Denote the corresponding Hilbert space and the norm by $L_\mu^2(\partial_+SD)$ and $\|\cdot\|_{L_\mu^2(\partial_+SD)}$, respectively. We will also write

$$h_\psi(x, v) = h(\phi_{-\tau(x, -v)}(x, v)), \quad (x, v) \in SD,$$

for $h \in C^\infty(\partial_+SD)$.

If F is a notation for a function space (C^k , L^p , H^k , etc.), then we will denote by $\mathcal{F}(D)$ the corresponding space of pairs $[f, \alpha]$ with f a function and α a 1-form on D . In particular, $\mathcal{L}^2(D)$ is the space of square integrable pairs $[f, \alpha]$, and we endow this space with the inner product

$$([f, \alpha], [f', \alpha'])_{\mathcal{L}^2(D)} = \int_D (ff' + \langle \alpha, \alpha' \rangle_{g_0}) d\text{Vol}_{g_0}.$$

Lemma 5.3. *If $f \in C^\infty(D)$, α is a smooth 1-form on D and $h \in C_0^\infty((\partial_+ SD)^{\text{int}})$, then*

$$(T_\lambda[f, \alpha], h)_{L^2_\mu(\partial_+ SD)} = ([f, \alpha], T_\lambda^* h)_{\mathcal{L}^2(D)},$$

where $T_\lambda^* h$ is a pair defined as

$$T_\lambda^* h(x) = \left[\int_{S_x D} h_\psi(x, v) e^{-\lambda\tau(x, -v)} d\sigma_x(v), \int_{S_x D} v^k h_\psi(x, v) e^{-\lambda\tau(x, -v)} d\sigma_x(v) \right].$$

Proof. By Santaló formula (see [21] or [2])

$$\begin{aligned} & (T_\lambda[f, \alpha], h)_{L^2_\mu(\partial_+ SD)} \\ &= \int_{\partial_+ SD} \int_0^{\tau(x, v)} [f(\gamma_{x,v}(t)) + \alpha_k(\gamma_{x,v}(t))\dot{\gamma}_{x,v}^k(t)] e^{-\lambda t} dt h(x, v) d\mu(x, v) \\ &= \int_{SD} [f(x) + \alpha_k(x)v^k] h_\psi(x, v) e^{-\lambda\tau(x, -v)} d\Sigma^{2n-3}(x, v) \\ &= \int_D f(x) \left(\int_{S_x D} h_\psi(x, v) e^{-\lambda\tau(x, -v)} d\sigma_x(v) \right) d\text{Vol}_g(x) \\ &\quad + \int_D \alpha_k(x) \left(\int_{S_x D} v^k h_\psi(x, v) e^{-\lambda\tau(x, -v)} d\sigma_x(v) \right) d\text{Vol}_g(x). \end{aligned}$$

This proves the statement. \square

Proof of Proposition 5.1. First, we extend (D, g_0) to a slightly larger simple manifold and to extend both f and α by zero. Then f and α are still in L^∞ , and in particular in L^p for all $1 < p < \infty$. In this way we can assume that both f and α are compactly supported in D^{int} .

We let b also depend on ω and change notations to write the assumption in the form

$$\int_{S_x D} \int_0^{\tau(x, v)} e^{-\lambda t} [f(\gamma_{x,v}(t)) + \alpha_k(\gamma_{x,v}(t))\dot{\gamma}_{x,v}^k(t)] b(x, v) dt d\sigma_x(v) = 0$$

for all $x \in \partial D$ and $b \in C_0^\infty((\partial_+ SD)^{\text{int}})$. Let \tilde{D} be a compact submanifold of D with boundary such that (\tilde{D}, g_0) is also simple, $\tilde{D} \subset\subset D^{\text{int}}$ and supports of f and α are compact subsets of \tilde{D}^{int} . Note that α is L^∞ on D (and in particular being in L^2 on D) implies that in particular $\delta\alpha \in H^{-1}(\tilde{D})$. Then we obtain the solenoidal decomposition $\alpha = \alpha^s + dp$ on \tilde{D} , where $\delta_{g_0}\alpha^s = 0$ and $p \in H_0^1(\tilde{D})$ with $-\Delta_{g_0}p = \delta\alpha$. Here $\delta_{g_0}\alpha = \nabla_{g_0}^i \alpha_i$, where ∇_{g_0} is the covariant derivative

corresponding to the metric g_0 . Extend p to D by zero, so that $p \in H_0^1(D)$ with $p = 0$ in $D \setminus \tilde{D}$. An integration by parts shows that we have

$$\int_{S_x D} \int_0^{\tau(x,v)} e^{-\lambda t} [f(\gamma_{x,v}(t)) + \lambda p(\gamma_{x,v}(t)) + \alpha_k^s(\gamma_{x,v}(t)) \dot{\gamma}_{x,v}^k(t)] b(x, v) dt d\sigma_x(v) = 0 \quad (16)$$

for all $x \in \partial D$ and $b \in C_0^\infty((\partial_+ SD)^{\text{int}})$. Next, we make the choice $b(x, v) = h(x, v)\mu(x, v)$ for $h \in C_0^\infty((\partial_+ SD)^{\text{int}})$ and integrate (16) over ∂D and get

$$(T_\lambda[f + \lambda p, \alpha^s], h)_{L^2_\mu(\partial_+ SD)} = 0.$$

We are now in the same situation as in the proof of Lemma 5.3, and using the Santaló formula implies

$$([f + \lambda p, \alpha^s], T_\lambda^* h)_{\mathcal{L}^2(D)} = 0$$

for all $h \in C_0^\infty((\partial_+ SD)^{\text{int}})$. Note that the last integral is absolutely convergent because $f \in L^\infty(D)$ and α is 1-form which is L^∞ on D , and also the previous steps are justified by Fubini's theorem.

It remains to choose $h = T_\lambda[\varphi, \beta]$ for $\varphi \in C_0^\infty(D^{\text{int}})$ and β being C_0^∞ -smooth 1-form in D^{int} , to obtain that

$$([f + \lambda p, \alpha^s], T_\lambda^* T_\lambda[\varphi, \beta])_{\mathcal{L}^2(D)} = 0.$$

Since $T_\lambda^* T_\lambda$ is self-adjoint, we have

$$(T_\lambda^* T_\lambda[f + \lambda p, \alpha^s], [\varphi, \beta])_{\mathcal{L}^2(D)} = 0$$

for all $\varphi \in C_0^\infty(D^{\text{int}})$ and for all C_0^∞ -smooth 1-form β in D^{int} . Therefore, $T_\lambda^* T_\lambda[f + \lambda p, \alpha^s] = 0$. By [10, Proposition 1], $T_\lambda^* T_\lambda$ is an elliptic pseudodifferential operator of order -1 in D^{int} . Here, ellipticity of $T_\lambda^* T_\lambda$ is in the sense that whenever f', α' are in $L^2(D^{\text{int}})$ and $T_\lambda^* T_\lambda[f', \alpha'] = 0$ and $\delta_{g_0} \alpha' = 0$, then f', α' are smooth. Since $f + \lambda p$ and α^s were compactly supported in D^{int} , this implies that $f + \lambda p$ and α^s are smooth and compactly supported in D^{int} . Hence $f + \lambda p$ and α^s are smooth in D and compactly supported in D^{int} . Now we can use the argument for smooth f and α given above, together with Proposition 5.2 to conclude that $f = -\lambda p - \lambda \psi$ and $\alpha = \alpha^s + dp = d\psi + dp$ for some $\psi \in C^\infty(D)$ with $\psi|_{\partial D} = 0$. To finish the proof, it remains to show that $p \in W^{1,\infty}(D)$. But this is clear from $dp = \alpha - \alpha^s$ and from α is L^∞ on D and α^s is C^∞ on D . \square

6. BOUNDARY DETERMINATION AND PROOF OF THEOREM 1.4

In this part we show boundary determination of the vector field X . For the generality of the statement we will assume the knowledge of the Cauchy data set $C_{g,X,q}$. It is easy to see that when 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g,X,q}$, knowledge of the Cauchy data set is equivalent to knowledge of the Dirichlet-to-Neumann map $N_{g,X,q}$. Moreover, we can determine not only the boundary values of X , but also the boundary values of q . This is the following proposition.

Proposition 6.1. *Let (M, g) be admissible, and let $m \geq 2$ be an integer. Suppose that X is a C^∞ vector field on M and $q \in C^\infty(M)$. Then the knowledge of the*

Cauchy data set $C_{g,X,q}$ *determines the boundary values of* X *and the boundary values of* q .

To prove this proposition, it suffices to show that for any $p \in \partial M$, $C_{g,X,q}^M$ determines $X(p)$ and $q(p)$. In the following we will consider this local problem.

Fix a point $p \in \partial M$ and let (x', x_n) be the boundary normal coordinates near p , where $x' = (x_1, \dots, x_{n-1})$. In these coordinates ∂M corresponds to $\{x_n = 0\}$, the vector field X becomes the differential operator $X = X^j \frac{\partial}{\partial x_j}$, and the metric tensor can be written as

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta + dx^n \otimes dx^n.$$

Here the in the following we use the convention that Greek indices run from 1 to $n-1$ and Roman indices from 1 to n . Denote $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, then the Laplace-Beltrami operator in the boundary normal coordinates takes the form

$$-\Delta_g = D_n^2 + iE(x)D_n + Q_2(x, D_{x'}) + Q_1(x, D_{x'}) \quad (17)$$

with E, Q_1, Q_2 given by

$$E(x) = \frac{1}{2} g_{\alpha\beta} \partial_n g^{\alpha\beta}, \quad (18)$$

$$Q_2(x, D_{x'}) = g^{\alpha\beta} D_\alpha D_\beta, \quad (19)$$

$$Q_1(x, D_{x'}) = -i(\frac{1}{2} g^{\alpha\beta} \partial_\alpha (\log |g|) + \partial_\alpha g^{\alpha\beta}) D_\beta. \quad (20)$$

Next we would like to write the $2m$ order equation

$$\mathcal{L}_{g,X,q} u = (-\Delta_g)^m u + Xu + qu = 0 \quad \text{in } M, \quad m \geq 2 \quad (21)$$

as a second order system. To this end, introduce

$$u_1 = u, \quad u_2 = (-\Delta_g)u, \quad \dots, \quad u_m = (-\Delta_g)^{m-1}u$$

and let $U = (u_1, \dots, u_m)^T$. By a standard reduction, (21) can be written as a system of equations in U :

$$\mathcal{L}_{A_{11}, A_{12}, A_0} U := (-\Delta_g \otimes I + iA_{11}(x, D_{x'}) + iA_{12}(x)D_n + A_0(x))U = 0 \quad \text{in } M, \quad (22)$$

where I is the $m \times m$ identity matrix, $A_{11}(x, D_{x'})$, $A_{12}(x)$ and $A_0(x)$ are defined by

$$A_{11}(x, D_{x'}) := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ X^\alpha(x)D_\alpha & 0 & \dots & 0 \end{pmatrix}, \quad A_{12}(x) := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ X^n(x) & 0 & \dots & 0 \end{pmatrix},$$

$$A_0(x) := \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ q(x) & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The associated Cauchy data set to the system (22) is

$$C_{A_{11}, A_{12}, A_0} := \{(U|_{\partial M}, \partial_\nu U|_{\partial M}) : \mathcal{L}_{A_{11}, A_{12}, A_0} U = 0 \text{ in } M, \quad U \in (H^2(M))^m\}.$$

It is easy to see that C_{A_{11}, A_{12}, A_0} and $C_{g, X, q}$ are mutually determined, hence it suffices to show C_{A_{11}, A_{12}, A_0} determines A_{11} , A_{12} and A_0 at $p \in \partial M$.

The following result gives a factorization of the operator $\mathcal{L}_{A_{11}, A_{12}, A_0}$. Similar techniques are employed in [3, 12, 16, 18].

Proposition 6.2. *There is a matrix-valued pseudodifferential operator $B(x, D_{x'})$ of order 1 in x' , depending smoothly on x_n , such that*

$$\mathcal{L}_{A_{11}, A_{12}, A_0} = (D_n \otimes I + iE(x) \otimes I + iA_{12}(x) - iB(x, D_{x'}))(D_n \otimes I + iB(x, D_{x'})) \quad (23)$$

modulo a smoothing operator. Moreover, the principle symbol of the operator $B(x, D_{x'})$ is $-\sqrt{Q_2(x, \xi')}I$. Here $E(x)$ and $Q_2(x, D_{x'})$ are given by (18) and (19) respectively.

Proof. Plug (17) into (22) we have

$$\begin{aligned} \mathcal{L}_{A_{11}, A_{12}, A_0} &= (D_n^2 + iE(x)D_n + Q_2(x, D_{x'}) + Q_1(x, D_{x'})) \otimes I \\ &\quad + iA_{11}(x, D_{x'}) + iA_{12}(x)D_n + A_0(x). \end{aligned}$$

Comparing this expression with (23) gives the following constrains on $B(x, D_{x'})$ modulo a smoothing operator:

$$\begin{aligned} &B^2(x, D_{x'}) + i[D_n \otimes I, B(x, D_{x'})] - E(x)B(x, D_{x'}) - A_{12}(x)B(x, D_{x'}) \\ &= Q_2(x, D_{x'}) \otimes I + Q_1(x, D_{x'}) \otimes I + iA_{11}(x, D_{x'}) + A_0(x). \end{aligned} \quad (24)$$

Let $b(x, \xi')$ be the full symbol of $B(x, D_{x'})$, then (24) implies on the level of symbols that

$$\sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_{x'}^\alpha b D_{\xi'}^\alpha b + \partial_n b - E(x)b - A_{12}(x)b = Q_2(x, \xi')I + Q_1(x, \xi')I + iA_{11}(x, \xi') + A_0(x). \quad (25)$$

Let $b \sim \sum_{j \leq 1} b_j$ where $b_j(x, \xi')$ is an $m \times m$ matrix with entries homogeneous of degree j in $\xi' = (\xi_1, \dots, \xi_{n-1})$. Collecting the terms homogeneous of degree 2 in (25) yields

$$b_1^2(x, \xi') = Q_2(x, \xi')I,$$

from which we can choose

$$b_1(x, \xi') = -\sqrt{Q_2(x, \xi')}I. \quad (26)$$

Collecting the terms homogeneous of degree 1 in (25) yields

$$b_0 b_1 + b_1 b_0 + \sum_{|\alpha|=1} \partial_{x'}^\alpha b_1 \partial_{\xi'}^\alpha b_1 + \partial_n b_1 - E(x) b_1 - A_{12}(x) b_1 = Q_1(x, \xi') I + i A_{11}(x, \xi'). \quad (27)$$

Since $b_1(x, \xi')$ has been determined above, $E(x)$ and $Q_1(x, \xi')$ are known from (18) (20), by some elementary linear algebra there exists a unique $b_0(x, \xi')$ satisfying this identity. Next collecting the terms of homogeneous of degree 0 in (25) implies

$$\begin{aligned} b_0^2 + b_1 b_{-1} + b_{-1} b_1 + \sum_{|\alpha|=1} \partial_{x'}^\alpha b_1 \partial_{\xi'}^\alpha b_0 + \sum_{|\alpha|=1} \partial_{x'}^\alpha b_0 \partial_{\xi'}^\alpha b_1 + \sum_{|\alpha|=2} \frac{1}{2} \partial_{x'}^\alpha b_1 \partial_{\xi'}^\alpha b_1 \\ + \partial_n b_0 - E(x) b_0 - A_{12}(x) b_0 = A_0(x), \end{aligned} \quad (28)$$

From which we can solve for $b_{-1}(x, \xi')$. In general, the term $b_j(x, \xi')$ can be determined by considering the terms homogeneous of degree $j+1$ in (25). This completes the proof. \square

Proof of Proposition 6.1. Using a similar argument as in [16, Proposition 1.2], we conclude that the Cauchy data set C_{A_{11}, A_{12}, A_0} determines the operator $B(x', 0, D_{x'})$ modulo a smoothing operator. Consequently each $b_j|_{x_n=0}$ is determined, $j \leq 1$. It follows from (27) that the following expression is determined by the Cauchy data set C_{A_{11}, A_{12}, A_0} :

$$-X^n|_{x_n=0} \sqrt{Q_2(x', 0, \xi')} + i X^\alpha|_{x_n=0} \xi_\alpha, \quad \xi' \in \mathbb{R}^{n-1}.$$

Varying ξ' determines $X^n|_{x_n=0}$ and $X^\alpha|_{x_n=0}$, $\alpha = 1, \dots, n-1$. Evaluating (28) on $\{x_n = 0\}$ shows that the Cauchy data set C_{A_{11}, A_{12}, A_0} determines $A_0|_{x_n=0}$, hence $q|_{x_n=0}$. \square

Proof of Theorem 1.4. If 0 is not a Dirichlet eigenvalue of $\mathcal{L}_{g, X_1, q_1}$ and $\mathcal{L}_{g, X_2, q_2}$, then $N_{g, X_1, q_1} = N_{g, X_2, q_2}$ implies $C_{g, X_1, q_1} = C_{g, X_2, q_2}$. By Proposition 6.1 we conclude that $X_1 = X_2$ on ∂M . The result then follows from Theorem 1.3. \square

7. PROOF OF THEOREM 1.5

We will follow the argument of [12, Theorem 1.3] and [3, Theorem 4]. Proceeding as in the proof of Theorem 1.3,

We can derive the following integral identity (see (13))

$$\int_M (X_{x_1}^b + i X_r^b) e^{i\lambda(x_1 + ir)} b(\theta) dr d\theta dx_1 = 0. \quad (29)$$

This is similar to (13) but this time the integral is over M instead of $\mathbb{R} \times M_{1, x_1}$ since we cannot extend the vector fields X_1 and X_2 any more. Varying the smooth function $b(\theta)$ leads to

$$\int_{M_\theta} (X_{x_1}^b + i X_r^b) e^{i\lambda(x_1 + ir)} d\bar{\rho} \wedge d\rho = 0 \quad \text{for all } \theta \in S^{n-2}$$

where $M_\theta := \{(x_1, r) \in \mathbb{R}^2 : (x_1, r, \theta) \in M\}$ and $\rho = x_1 + ir$. Define

$$f(x') = \int_{\mathbb{R}} e^{i\lambda x_1} X_{x_1}^\flat(x_1, x') dx_1, \quad \alpha(x') = \sum_{j=2}^n \left(\int_{\mathbb{R}} e^{i\lambda x_1} X_j^\flat(x_1, x') dx_1 \right) dx^j.$$

Here we extend X as zero outside of M so that the integral in x_1 can be over \mathbb{R} . The above argument shows that

$$\int e^{-\lambda r} [f(\gamma_{\omega, \theta}(r)) + i\alpha(\dot{\gamma}_{\omega, \theta}(r))] dr = 0 \quad \text{for all } \theta \in S^{n-2},$$

where the r -integrals are integrals over geodesics $\gamma_{\omega, \theta}$ in $\pi(M) \subset M_0$. Observe that $\alpha(x')$ is $W^{1, \infty}$ on $\pi(M)$ and $f(x')$ is L^∞ on $\pi(M)$. Under the assumption that $(\pi(M), g_0)$ is a simple $(n-1)$ -dimensional manifold, we can apply Proposition 5.1 to $D := \pi(M)$ and conclude that for small enough λ , we have $f = -\lambda p$ and $\alpha = -idp$ where $p \in W^{1, \infty}(\pi(M))$ and $p|_{\partial\pi(M)} = 0$. The definition of α and analyticity of the Fourier transform imply that

$$\partial_k X_j^\flat - \partial_j X_k^\flat = 0, \quad j, k = 2, \dots, n \quad \text{in } M^{\text{int}}.$$

Also

$$\int e^{i\lambda x_1} (\partial_j X_1^\flat - \partial_1 X_j^\flat)(x_1, x') dx_1 = \partial_j f + i\lambda \alpha_j = 0 \quad \text{in } M^{\text{int}},$$

showing that $dX^\flat = 0$ in M . Since M is simply connected, there exists a function ϕ such that $\nabla\phi = X \in W^{1, \infty}(M)$. By [9, Theorem 4.5.12 and Theorem 3.1.7] we have $\phi \in C^{1,1}(M)$.

Next we need to show that ϕ is constant on ∂M . In the case where we can extend X to be a compactly supported $W^{1, \infty}$ vector field on a larger manifold, that ϕ is constant on ∂M simply follows from the construction, but here we have to prove it. This is the content of the next proposition.

Proposition 7.1. *The function ϕ is constant on the connected boundary ∂M .*

Proof. Let us start by constructing more complex geometric optics solutions. From Proposition 3.1, Remark 3.2 and Remark 3.3 we can choose complex geometric optics solutions of the form

$$\begin{aligned} u &= e^{-\frac{1}{h}(x_1+ir)} (|g|^{-1/4} c^{1/2} a_0(x_1, r, \theta) b(\theta) + hr_1), \\ v &= e^{\frac{1}{h}(x_1+ir)} (|g|^{-1/4} c^{1/2} + hr_2), \end{aligned}$$

where $\bar{\partial}a_0 = 0$, $\lambda \in \mathbb{R}$ and $\|r_j\|_{H_{\text{sel}}^1(M_1)} = \mathcal{O}(1)$, $j = 1, 2$. Note that in the previous construction we choose $a_0 = e^{i\lambda(x_1+ir)}$ but this time we need more a_0 's. Substituting these solutions in (12), multiplying the resulting equality by h and letting $h \rightarrow 0$, we get

$$\lim_{h \rightarrow 0} \int_M \langle X, \nabla\rho \rangle_g uv d\text{Vol}_g(x) = 0,$$

where $\rho = x_1 + ir$ and $X = X_1 - X_2$. Recall that $X := \nabla\phi$. Insert the above complex geometric optics solutions yields

$$\int_M \bar{\partial}\phi a_0(x_1, r, \theta) b(\theta) dr d\theta dx_1 = 0.$$

Varying the smooth function $b(\theta)$ leads to

$$\int_{M_\theta} \bar{\partial}\phi a_0 d\bar{\rho} \wedge d\rho = 0 \quad \text{for all } \theta \in S^{n-2}.$$

Integrating by parts and using that $\bar{\partial}a_0 = 0$ gives

$$\int_{\partial M_\theta} \phi a_0 d\rho = 0 \quad \text{for all } \theta \in S^{n-2} \tag{30}$$

and for every a_0 with $\bar{\partial}a_0 = 0$.

On the other hand, noticing that in solving the eikonal equation (7), we may choose $\varphi = x_1$ but $\psi = -r$. Then we can construct complex geometric optics solutions of the form

$$\begin{aligned} u &= e^{-\frac{1}{h}(x_1 - ir)} (|g|^{-1/4} c^{1/2} a_0(x_1, r, \theta) b(\theta) + hr_1), \\ v &= e^{\frac{1}{h}(x_1 - ir)} (|g|^{-1/4} c^{1/2} + hr_2), \end{aligned}$$

where $\partial a_0 = 0$, $\lambda \in \mathbb{R}$ and $\|r_j\|_{H_{\text{scl}}^1(M_1)} = \mathcal{O}(1)$, $j = 1, 2$. Here

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial r} \right).$$

Using a similar argument as in the preceding paragraph we can derive

$$\int_{\partial M_\theta} \phi \tilde{a}_0 d\bar{\rho} = 0 \quad \text{for all } \theta \in S^{n-2}$$

and for every \tilde{a}_0 with $\bar{\partial}\tilde{a}_0 = 0$. In particular we can choose $\tilde{a}_0 = \bar{a}_0$ where a_0 solves $\bar{\partial}a_0 = 0$. Then taking complex conjugate gives

$$\int_{\partial M_\theta} \bar{\phi} a_0 d\rho = 0 \quad \text{for all } \theta \in S^{n-2}. \tag{31}$$

Combining (30) and (31) we see

$$\int_{\partial M_\theta} \operatorname{Re} \phi a_0 d\rho = 0, \quad \int_{\partial M_\theta} \operatorname{Im} \phi a_0 d\rho = 0$$

for all a_0 with $\bar{\partial}a_0 = 0$. Using the argument in [4, Section 5] implies that $\operatorname{Re} \phi|_{\partial M_\theta} = F|_{\partial M_\theta}$ for some non-vanishing holomorphic function F on M_θ . Observing that $\operatorname{Im} F$ is a harmonic function in M_θ and $\operatorname{Im} F|_{\partial M_\theta} = 0$, we conclude that F is real-valued and hence is constant on each connected component of ∂M_θ . Varying θ shows that $\operatorname{Re} \phi$ is constant along ∂M . Likewise we can show $\operatorname{Im} \phi$ is also constant along ∂M . \square

Proof of Theorem 1.5. Since $\phi = c$ for some constant c along ∂M , replacing ϕ by $\phi - c$ if necessary, we may assume $\phi = 0$ on ∂M . The rest part of the proof is the same as that of Theorem 1.3. \square

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